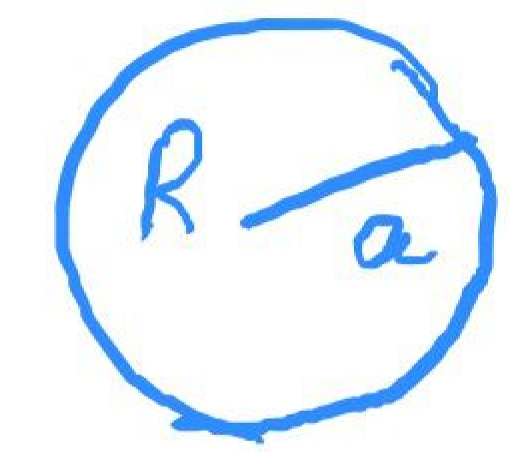
wave eqn. for disk:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

$$u\big|_{\partial R} = 0$$



radius = a

have found solutions of form

$$u(r,\theta,t) = f(r)\, g(\theta)\, h(t)$$

where $f(r) = J_m(\sqrt{\lambda_{mn}}\, z)$ with $\lambda_{mn} = \frac{z_{mn}^2}{a^2}$

z_{mn} = nth zero of J_m

$$g(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

$$h(t) = a_{mn} \cos \sqrt{\lambda_{mn}}\, ct + b_{mn} \sin \sqrt{\lambda_{mn}}\, ct$$

See: frequencies related to zeros of Bessel functions J_m, $m \geq 0$

Bessel's equation:

$$z^2 \frac{\partial^2 f}{\partial z^2} + z \frac{\partial f}{\partial z} + (z^2 - m^2) f = 0$$

one can show:

general solution given by $c_1 J_m(z) + c_2 Y_m(z)$

Bessel function of first kind, $m \geq 0$ — second kind

$|J_m(0)| < \infty \quad \forall m \geq 0 \qquad |Y_m(z)| \to \infty$ for $z \to 0$

The $J_m(z)$'s can be expressed via power series

$$J_m(z) = \sum_{j=0}^{\infty} a_j z^{j+m}$$

(please ignore what I wrote for $Y_m(z)$ last time)

($Y_m(z)$'s involve $\log z$)

Coefficients can be calculated recursively by plugging power series into DE.

$\Rightarrow$ gets that $a_1 = 0$

and $\boxed{a_j = \frac{1}{j(j+2m)} a_{j-2}}$ $\Big\}$ $a_{odd} = 0$

usually normalized such that

$$\boxed{J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+m}}{2^{2k+m} k! (2m+k)!}}$$

converges for all z's !

Qualitative discussion of Bessel functions J_m

they are solutions of ODE

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0 \qquad \frac{1}{z^2}$$

$$\frac{d^2 f}{dz^2} = -\left(1 - \frac{m^2}{z^2}\right) f - \frac{1}{z} \frac{df}{dz}$$

$$\left(1 - \frac{m^2}{z^2}\right) f \to f, \qquad \frac{1}{z}\frac{df}{dz} \to 0 \qquad \text{for } z \to \infty$$

$$\text{i.e.} \quad -\left(1 - \frac{m^2}{z^2}\right) f \to -f$$

equation can be interpreted as equ. for oscillator with friction force $\frac{df}{dz}$

$\Longrightarrow$ expect oscillations with amplitudes becoming smaller can show:

for large z $\quad J_m(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$

zeros of $\cos z$ $\quad = \quad (n+\frac{1}{2}\pi)$

$$n+\frac{1}{2}\pi \approx z - \frac{\pi}{4} - m\frac{\pi}{2}$$

$$\Rightarrow \quad z \approx \left(s - \frac{1}{4}\right)\pi \qquad s \text{ an integer}$$

some numbers from book (Section 7.8)

m	z_{on}	exact value	simple approx. $(s-\frac{1}{4})\pi$	percentage error
1	z_{01}	2.40483	2.35619	2%
2	z_{02}	5.52008	5.49779	.4%
3	z_{03}	8.65373	8.63938	.2%

<u>BUT</u> $\quad J_m(0) = 0$ for $m > 0$. $\quad J_0(0) = 1$

again: eigenvalues given by zeros of Bessel function

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

Consider special case with circular symmetry

i.e. initial value functions only depend on r, not on θ

$$u(r,\theta,0) = \alpha(r)$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r)$$

$$\left(\Rightarrow \quad \frac{\partial^2 u}{\partial \theta^2} = 0 \right)$$

$\Rightarrow$ get PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{1}{r}\,\partial/\partial r\left(r\,\frac{\partial u}{\partial r}\right)\right)$$

again consider product sol. $\boxed{u(r,t) = f(r)\,h(t)}$

separate variables

$$\frac{1}{c^2 h(t)} h''(t) = \frac{1}{r f(r)} \frac{d}{dr}\left(r \frac{d f(r)}{dr}\right) = -\lambda \quad | r^2$$

get as before:

$$\boxed{\begin{aligned} h''(t) &= -\lambda c^2 h(t) \\ r^2 f''(r) + r f'(r) + \lambda r^2 f(r) &= 0 \end{aligned}}$$

doing substitution $z = \sqrt{\lambda}\, r$

we get special case of Bessel's equation $(m=0)$

$$\boxed{z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + z^2 f = 0}$$

relevant solution: Bessel function $J_0(z)$ of first kind.

$\Rightarrow$ General solution given by

$$u(r,t) = \sum_{n=1}^{\infty} \underbrace{J_0(\sqrt{\lambda_{0n}}\, r)}_{f(r)} \underbrace{(a_n \cos\sqrt{\lambda_{0n}}\, ct + b_n \sin\sqrt{\lambda_{0n}}\, ct)}_{h(t)}$$

Problem: how to calculate coefficients a_n and b_n ?

use initial conditions:

$$\boxed{\begin{aligned} \alpha(r) &= u(r,0) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}\, r)\, a_n \\ \beta(r) &= \frac{\partial u}{\partial t}(r,0) = \sum_{n=1}^{\infty} J_0(\sqrt{\lambda_{0n}}\, r)\, \sqrt{\lambda_{0n}}\, c b_n \end{aligned}}$$